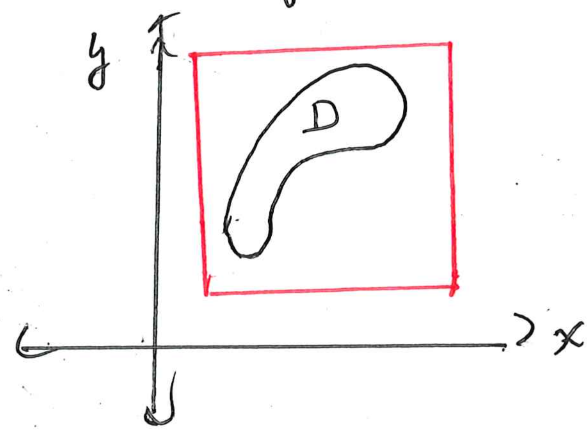
PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON TUESDAY, APRIL 16, 2024, ON SECTION 15.2 – THE DOUBLE INTEGRAL OVER MORE GENERAL REGIONS

CLASS # 25

Sec 15.2: Defining Double Integrals over more general regions D.

Let $z = f(x,y)$ be given. Suppose D is a Closed and Bounded Region in the x,y plane that is contained in the Domain of f.



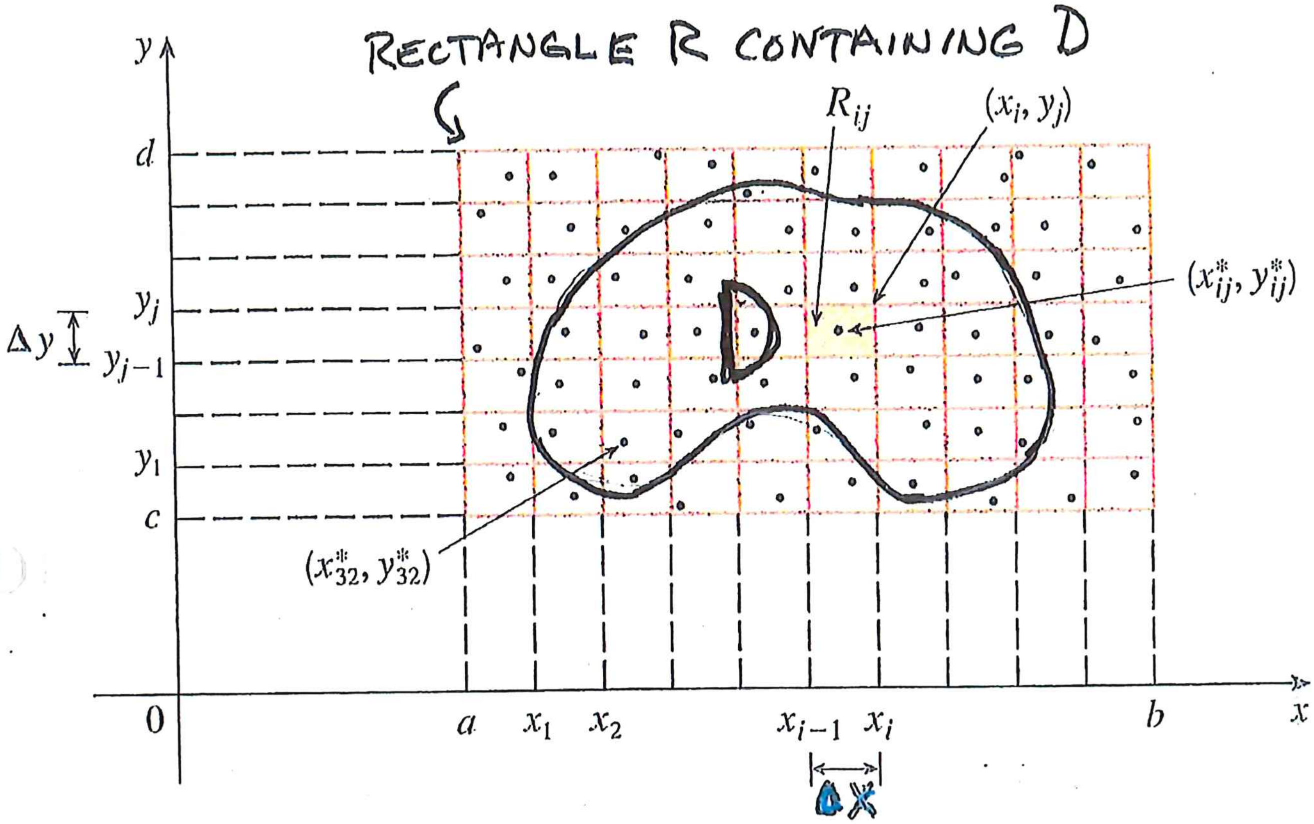
Dr. Shirley's Defn of the Double Integral over Region D

$$\iint_D f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \Big(\underbrace{\sum_{i's} \sum_{j's} f(x_{ij}^*, y_{ij}^*)\,\Delta A_{ij}}_{} \Big)$$

Only for those selected points (x_{ij}^*, y_{ij}^*), such that (x_{ij}^*, y_{ij}^*) is in Region D.

How do we determine what number this is?

$$\iint_D f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \left(\underbrace{\sum_{i\text{'s}} \sum_{j\text{'s}}}_{\text{where the restrictions below hold.}} f(x^*_{ij}, y^*_{ij})\,\Delta A_{ij} \right)$$



HERE, the Double Riemann Sum $R_{m,n}$

is $R_{m,n} = \underbrace{\sum_{i\text{'s}} \sum_{j\text{'s}}}_{} f(x^*_{ij}, y^*_{ij})\,\Delta A_{ij}$

Only for those selected points (x^*_{ij}, y^*_{ij}) such that (x^*_{ij}, y^*_{ij}) is in Region D.

15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

p. 1052

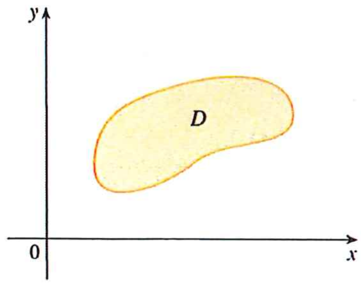


FIGURE 1

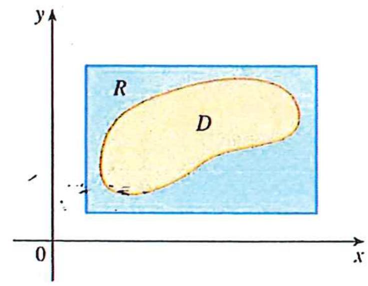


FIGURE 2

If F is integrable over R, then we define the **double integral of f over D** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined in Section 15.1. The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geqslant 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

p. 1052

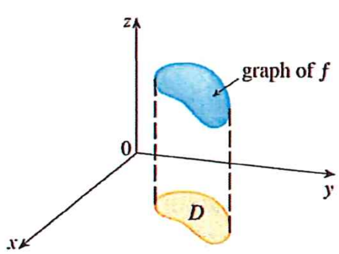


FIGURE 3

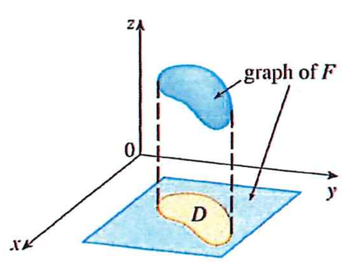


FIGURE 4

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D. Nonetheless, if f is continuous on D and the boundary curve of D is "well behaved" (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x, y)\, dA$ exists

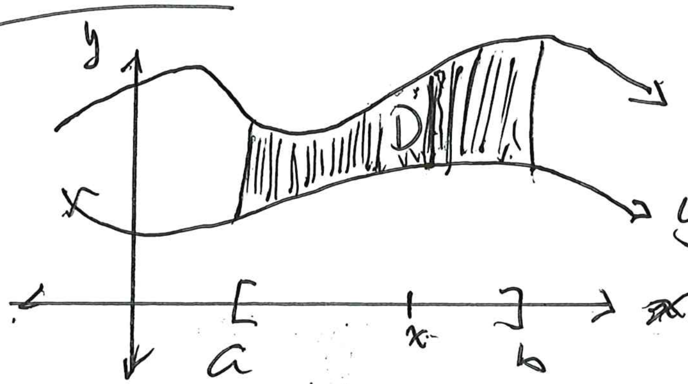
We can use iterated integrals —

(1) If the Region D is a Type I Region.

OR

(2) If the Region D is a Type II Region

Type I Region



For all $a \le x \le b$

$g_1(x) \le g_2(x)$

Type I Description of D:

$D = \{(x,y)$ such that $a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

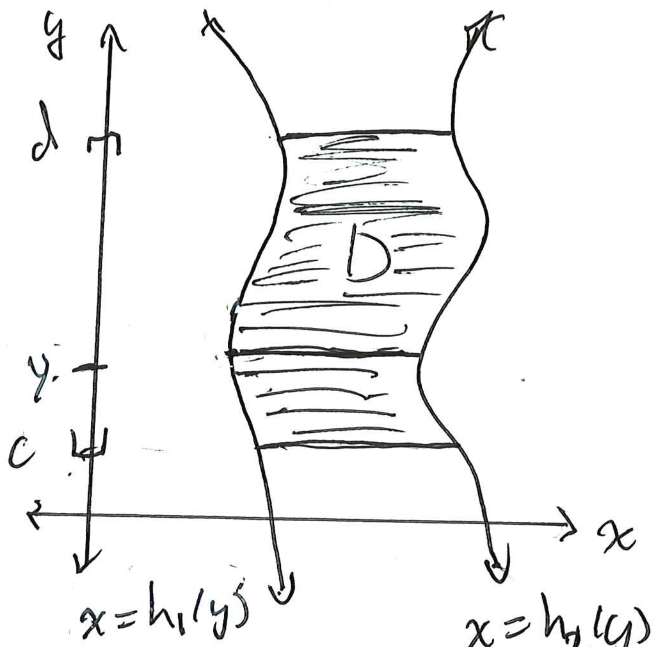
SHORTHAND: $D: a \le x \le b$ and $g_1(x) \le y \le g_2(x)$.

$$\iint_D f(x,y)\,dA = \int_a^b \underbrace{\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy}\,dx$$

$dy\,dx \longleftrightarrow$ Type I Region

Type II Region

For all y with $c \le y \le d$, $h_1(y) \le h_2(y)$



A Type II Description of D

$D = \{(x,y)$ such that $c \le y \le d$ and $h_1(y) \le x \le h_2(y)\}$

$D: c \le y \le d$ and $h_1(y) \le x \le h_2(y)$.

$$\iint_D f(x,y)\,dA = \int_c^d \underbrace{\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx}\,dy$$

$dx\,dy \Longleftrightarrow$ Type II Region

2

p.1053

and therefore $\iint_D f(x, y)\,dA$ exists. In particular, this is the case for the following two types of regions.

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

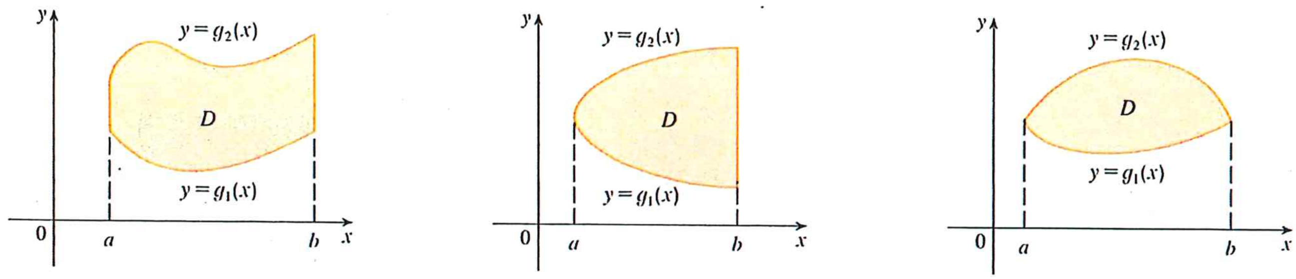


FIGURE 5
Some type I regions

In order to evaluate $\iint_D f(x, y)\,dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D. Then, by Fubini's Theorem,

$$\iint_D f(x, y)\,dA = \iint_R F(x, y)\,dA = \int_a^b \int_c^d F(x, y)\,dy\,dx$$

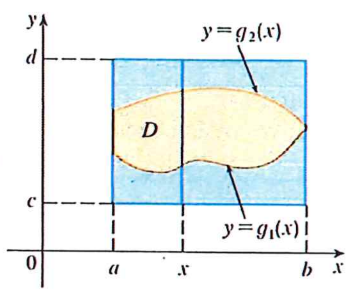


FIGURE 6

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\,dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \leqslant y \leqslant g_2(x)$. Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

3 If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx$$

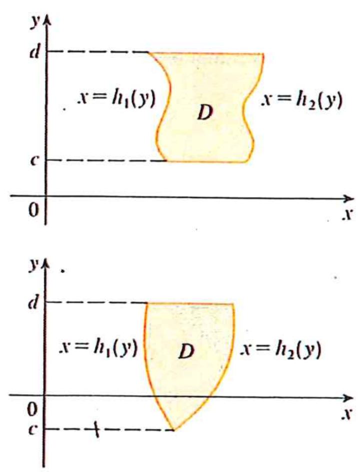


FIGURE 7
Some type II regions

p.1054

The integral on the right side of (3) is an iterated integral that is similar to the ones we considered in the preceding section, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

We also consider plane regions of **type II**, which can be expressed as

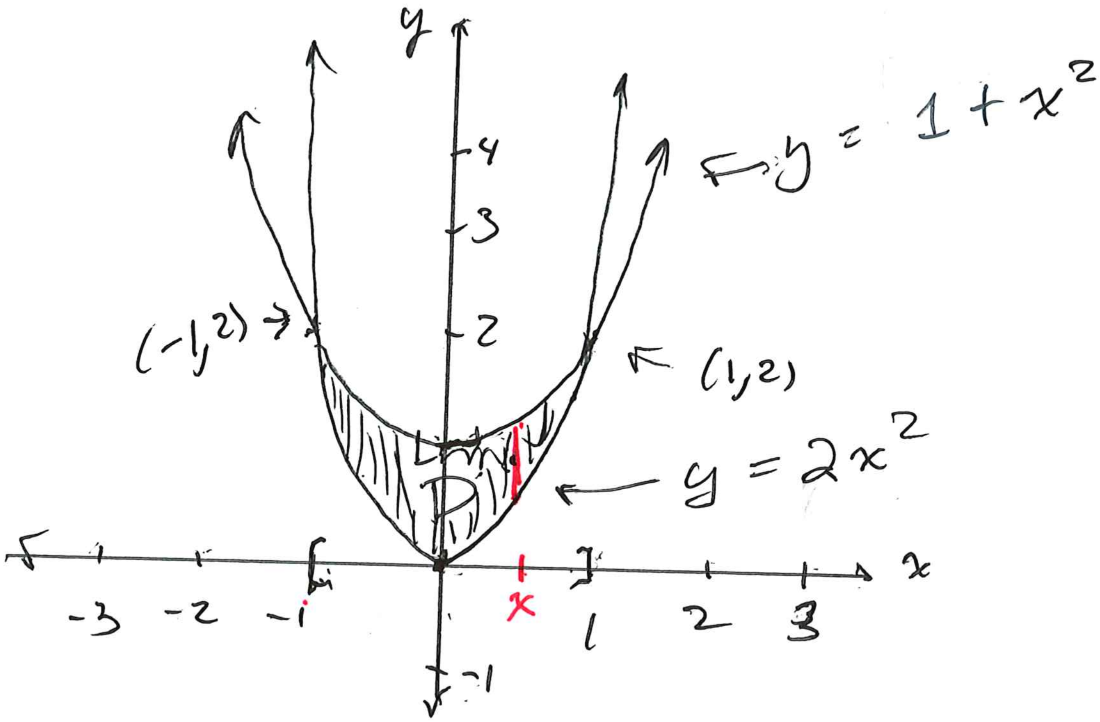
4
$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

A TYPE I EXAMPLE

Problem: Determine $\iint_D (x+2y)\,dA$

where D is the region bounded by $y = 2x^2$ and $y = 1 + x^2$.



Type I Description of D: $-1 \le x \le 1$ and $2x^2 \le y \le 1 + x^2$

$y = 2x^2$

$$\iint_D (x+2y)\,dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} \left((xy + y^2) \Big|_{y=2x^2}^{y=1+x^2} \right) dx$$

$$= \int_{-1}^{1} \left(\left(x(1+x^2) + (1+x^2)^2 \right) - \left(x(2x^2) + (2x^2)^2 \right) \right) dx$$

$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1 \right) dx$$

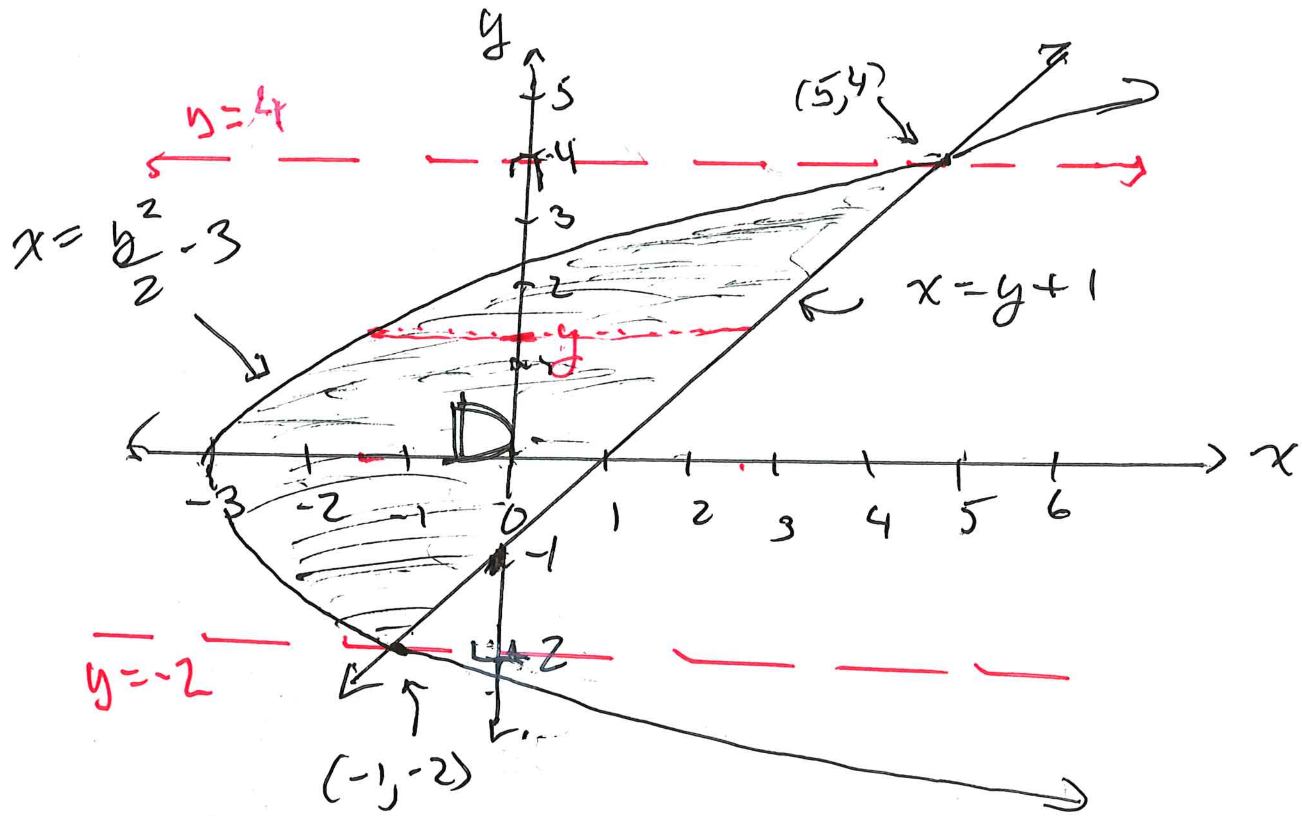
$$= \left(-\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right) \Big|_{-1}^{1}$$

$$= \cdots = \frac{32}{15} = 2^{2}/_{15}$$

$$\iint_D (x+2y)\,dA = \frac{32}{15} = 2^{2}/_{15}$$

A Type II EXAMPLE:

Find $\iint_D (xy)\,dA$ when D is the region bounded by $x = \frac{y^2}{2} - 3$ and $x = y+1$



A Type II Desc of D: $-2 \leq y \leq 4$ and

$$\left(\frac{y^2}{2} - 3\right) \leq x \leq (y+1)$$

$$\iint_D xy\,dA = \int_{-2}^{4} \left(\int_{x=\frac{y^2}{2}-3}^{y+1} xy\,dx \right) dy$$

$$= \dots = 36$$

Sometimes, Integration Requires you to switch the order of integration

Problem: (A) Sketch the Region D that this iterated integral evaluates the Double Integral of $z = \sin(y^2)$ over:

For $f(x,y) = \sin(y^2) + 0x = \sin y^2$,

$$\int_0^1 \int_x^1 \sin(y^2)\, \underline{dy\, dx} = \iint_D \sin(y^2)\, dA$$

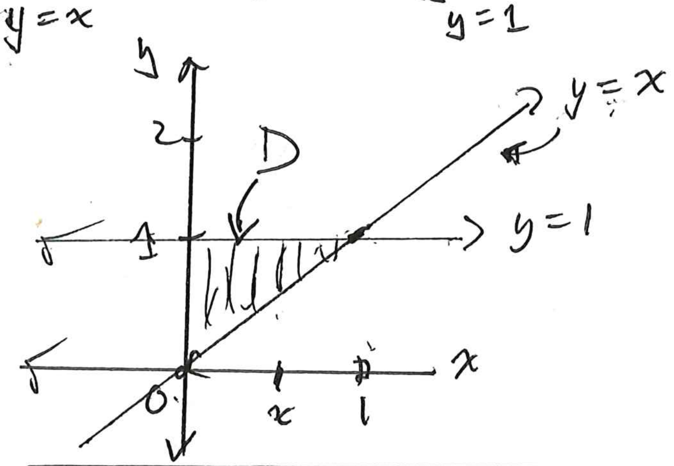
Type I

Type I Description of D

$D: 0 \le x \le 1$

$x \le y \le 1$

$y = x$ $\quad$ $y=1$



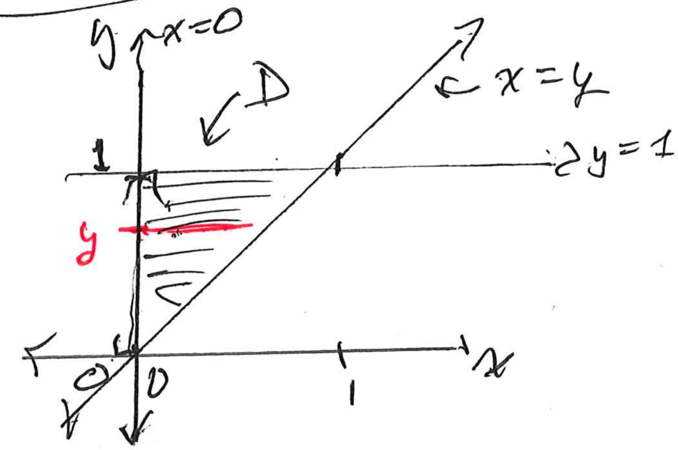
(B) Determine $\iint_D \sin(y^2)\, dA$

$$\iint_D \sin(y^2)\, dA = \int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

We must switch the order of Integration from $dy\,dx$ to $dx\,dy$.

A Type II Desc of D

$D: 0 \le y \le 1$ and $0 \le x \le y$



$$\iint_D \sin(y^2)\, dA = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \left(\left(x \sin(y^2) \right) \Big|_0^y \right) dy$$

$$= \int_0^1 \left(y \sin(y^2) - (0) \right) dy$$

$$= \int_0^1 y \sin(y^2)\, dy$$

$u = y^2,$
$du = 2y\,dy$
$y\,dy = \frac{1}{2} du$
when $y = 0$, $u = 0$
& when $y = 1$, $u = 1$

$$= \frac{1}{2} \int_0^1 \sin(u)\, du$$

$$= \dots = \frac{1}{2}(1 - \cos 1)$$

$$\iint_D \sin(y^2)\, dA = \frac{1}{2}(1 - \cos 1)$$

Problem: Change the Order of Integration.

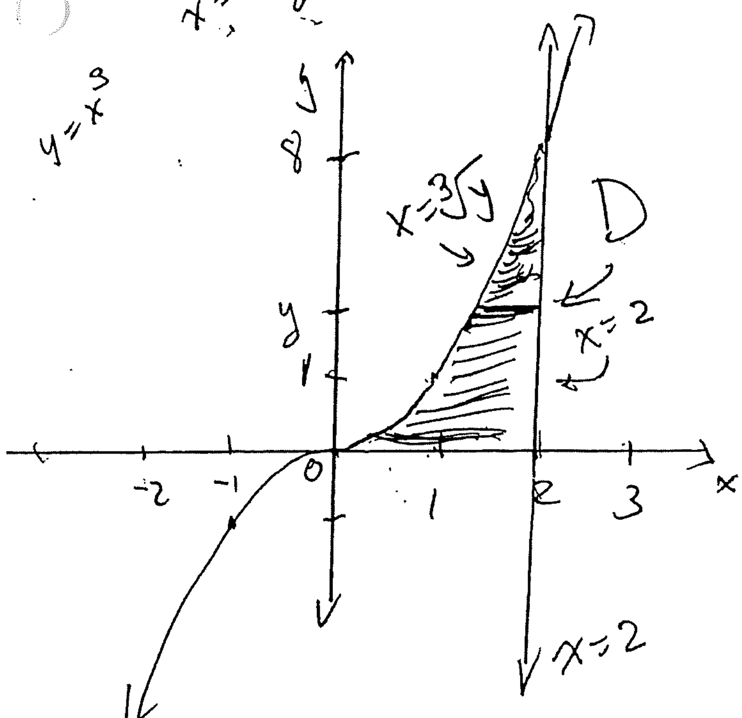
$$\iint_D f(x,y)\,dA = \int_0^8 \int_{x=\sqrt[3]{y}}^{x=2} f(x,y)\,dx\,dy$$

↖ Type II Region

A Type II Desc. of D:

$D: \; 0 \le y \le 8$

$\quad\; \underset{x=}{\sqrt[3]{y}} \le x \le \underset{x=}{2}$

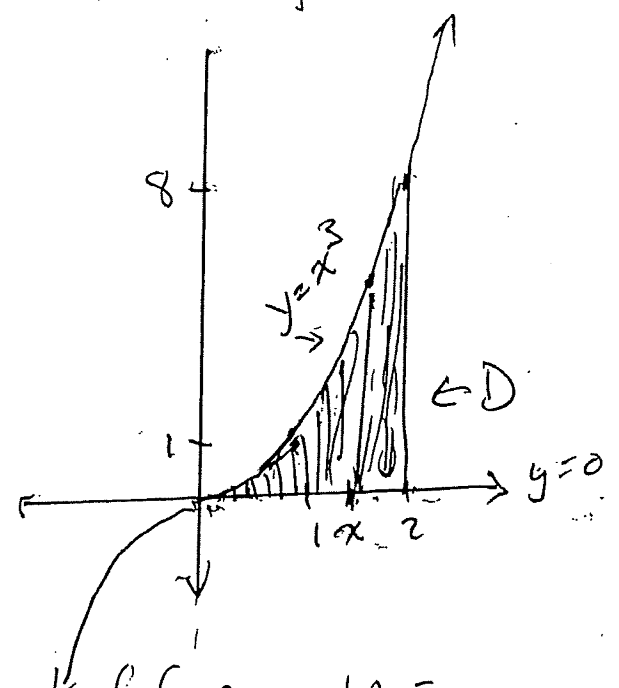


Switch the order from dxdy to dy dx

A Type I Desc of D

$D: \; 0 \le x \le 2$

$\quad\; \underset{y=}{0} \le y \le \underset{y=}{x^3}$



$$\iint_D f(x,y)\,dA =$$

$$\int_0^2 \int_0^{x^3} f(x,y)\,dy\,dx$$